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MODELS OF ELASTIC MEDIA WITH STRESS RELAXATION†

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The stress relaxation process is linked to the change with time of the metric stress tensor of the medium. Possible types of thermodynamically justified relaxation equations are discussed.

1. THE EQUATIONS OF THE THEORY OF ELASTICITY

AN ELASTIC medium is a special case of a simple medium, i.e. of a medium whose state can be described by three functions $x^i(\xi^a, t)$ of the law of motion and one thermodynamic function represented here by the entropy density per unit mass $S(\xi^a, t)$.

We will adopt the notation used in [1]: x^i are the coordinates in a three-dimensional Euclidean observer space and ξ^a in Lagrangian space, letters $i, j, k, \ldots$ from the middle of the Latin alphabet denote the projections of the tensors on the basis vectors of Eulerian space, the letters $a, b, c, d, \ldots$ from the beginning of the Latin alphabet denote the projections on the basis vectors of Lagrangian space, $x_a{}^i = \partial x^i(\xi^b, t)/\partial \xi^a$, $\xi_i{}^a = \partial \xi_i{}^a(x^i, t)/\partial x$ is the direct and reverse distortion, g_{ij}, $g^{ij} = \|g_{ij}\|^{-1}$ are the covariant and contravariant components of the metric tensor in Euclidean observer space, the Eulerial indices $i, j\ k, \ldots$ are juggled using the metric g_{ij}, repeated upper and lower indices denote summation, the indices $a, b, c, d, \ldots$ are juggled using the metric tensor of the medium in the Lagrangian system of coordinates $g_{ab} = x_a{}^i x_b g_{ij}$, $g^{ab} = \xi_i{}^a \xi_j{}^b g^{ij}$ and the quantities referring to initial state are denoted by a zero superscript $g_{ab}{}^0$, $x_a{}^{0i}$, ρ^0, $S^0, \ldots$; $d/dt = \partial/\partial t + \nabla^i \partial/\partial x^i$ is the tensor time derivative for constant Lagrangian coordinates, $v^i = dx^i(\xi^a, t)/dt$ are the components of the velocity vector, p^i, $p^{ab} = \xi_i{}^a \xi_j{}^b p^{ij}$, $p_b{}^a = p^{ac} g_{bc}$ are the components of the Cauchy stress tensor projected on different bases, U is the internal energy density of the medium per unit mass, T is the absolute temperature, q^i, $q^a = q^i \xi_i{}^a$ are the components of the thermal flux vector, and ∇_i, $\nabla^i = g^{ij}\nabla_j = \nabla_j g^{ij}$ and $\nabla_a = x_a{}^i \nabla_i = \nabla_i x_a{}^i$, $\nabla^a = g^{ab}\nabla_b = \nabla_b g^{ab}$ are the operators of covariant differentiation over the metric of the Eulerian and Lagrangian space, respectively.

The system of equations of the mechanics of simple media consists of the equations of momenta energy and entropy balance

$$\begin{gathered}\rho\frac{dv^i}{dt} = \nabla_j p^{ij} + \rho f^i \\ \rho\frac{dU}{dt} = p^{ij}\nabla_j v_i - \nabla_i q^i + r \\ \rho\frac{dS}{dt} + \nabla_i\left(\frac{q^i}{T}\right) - \frac{r}{T} = \left[q^i \nabla_i \frac{1}{T} + \frac{\rho}{T}\, q'\right] \equiv \sigma\end{gathered} \tag{1.1}$$

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Here f^i is the density of external volume forces, r is the external volume heat source, ρ is the density of the medium given explicitly by the law of motion and density of the initial state by the formula

$$\rho = \rho^0(\xi^a)\sqrt{g^0(\xi^a)}/\sqrt{g(\xi^a, t)}$$
$$g^0 = \det \| g_{ab}{}^0 \|, \quad g = \det \| g_{ab} \| \tag{1.2}$$

q' is the uncompensated heat and σ is the increase in entropy, which by virtue of the second law of thermodynamics must be non-negative.

In order to close the system of equations (1.1) we must specify the equations of state, which express the quantities U, p^{ij}, q^i and q' in terms of four basic unknown functions $x^i(\xi^a, t)$ and $S(\xi^a, t)$.

In the theory of elasticity the equations of state are defined if the dependence of internal energy density U on the components of metric tensor of the medium g_{ab} and entropy S is known

$$U = U(g_{ab}, S), \quad p^{ab} = 2\rho\partial U/\partial g_{ab}, \quad T = \partial U/\partial S \tag{1.3}$$

$$q' = 0 \tag{1.4}$$

The Fourier law of heat conduction

$$q^i = -\varkappa^{ij}\nabla_j T \tag{1.5}$$

where $\varkappa$ is the symmetrical thermal conductivity tensor, can be used as the relation for the thermal flux vector $\mathbf{q}$.

The system of equations (1.1), taking (1.2)–(1.4) into account, consists of five relations for four unknown functions $x^i(\xi^a, t)$ and $S(\xi^a, t)$. The system is, however, not overdefined, since the equations of energy and entropy balance [the last two equations of (1.1)] are interdependent by virtue of the equations of state (1.3) and (1.4). Subtracting the third equation of (1.1) from the second equation of (1.1) multiplied by T, we find that

$$dU/dt = \rho^{-1}p^{ij}\nabla_j v_i + TdS/dt - q' \tag{1.6}$$

We can confirm the validity of identity (1.6) by direct substitution of Eqs (1.3) and (1.4), and we make use here of the kinematic identity [1]

$$dg_{ab}/dt = \nabla_a v_b + \nabla_b v_a = x_a{}^i x_b{}^j (\nabla_i v_j + \nabla_j v_i)$$

Relation (1.6) represents one of the expressions for the basic thermodynamic Gibbs identity. The interdependence of Eqs (1.1) and the existence of their resulting identity represent, by virtue of the equations of state, together with the condition that the increase in entropy σ is non-negative, the laws restricting the possible form of the equations of state.

When the medium is made more complex by adding new functions defining its state, then every freshly introduced characteristic requires that an equation be constructed for it. The additional equations together with system (1.1) and the equations of state, must again admit of the existence of an identity corollary and ensure the non-negativity of the production of entropy σ.

2. THE RELAXATION EQUATIONS IN AN ISOTROPIC MEDIUM

The scalar form of the function $U(g_{ab}, S)$ implies the existence, amonst its arguments, of additional tensor quantities [2]. In the isotropic case, such additional tensor arguments can be represented by the components of the metric tensor (MT) of the standard, stress-free state (SSS), realized at some fixed standard value of the entropy S_*. The components of the MT SSS will be henceforth denoted by η_{ab} (the covariant components) and $\eta^{ab} = \|\eta_{ab}\|^{-1}$ (the contravariant components), and the values of all the remaining quantities in the SSS will be marked with an asterisk.

In the theory of elasticity the SSS is usually identical with the initial state, $\eta_{ab} = g_{ab}{}^0 = x_a{}^{0i} x_b{}^{0j} g_{ij}$, and

$$d\eta_{ab}/dt = 0 \tag{2.1}$$

Moreover, $\eta_{ab} \neq g_{ab}{}^0$ and the medium in question becomes more complicated by virtue of allowing a change with time of the components of the MT SSS $\eta_{ab}(\xi^a, t)$ which, together with the functions $x^i(\xi^a, t)$ and $S(\xi^a, t)$, become the parameters describing the state of the medium. The equations describing the change of MT η_{ab} with time will be called the relaxation equations. We shall now take, as the deformation of the medium, the difference between the MT g_{ab} of the actual state of the medium and the MT η_{ab} of the SSS. This deformation is often called, in the literature, the elastic strain so as to distinguish it from the plastic deformation with which the difference between the MT η_{ab} and the MT of the medium in its initial state $g_{ab}{}^0$ is associated. We shall call here the difference between g_{ab} and $g_{ab}{}^0$ the total deformation. From the physical point of view, it will perhaps be more natural to regard as the deformation the difference between the actual state of the medium and the SSS, i.e. the difference between g_{ab} and η_{ab}, and regard the difference between η_{ab} and $g_{ab}{}^0$ as the evolution of the internal state of the medium. When such terminology is used, the stress relaxation process will be accompanied by a change in the deformations, although no external motion of the medium may be apparent. Therefore the terms "relaxation of stresses" and "relaxation of deformations" refer to the same single process.

We can adopt, without loss of generality, the relaxation equations in the form

$$d\eta_{ab}/dt = 2\varphi_a{}^c\eta_{cb} \tag{2.2}$$

where the components of the relaxation tensor φ must be specified as functions of the defining parameters.

Note that the spherical part of the tensor φ is responsible for the change in the density of the SSS. Indeed,

$$\rho_* \frac{d}{dt}\frac{1}{\rho^*} = \frac{\rho^0\sqrt{g^0}}{\sqrt{\eta}}\frac{d}{dt}\frac{\sqrt{\eta}}{\rho^0\sqrt{g^0}} = \frac{1}{2\eta}\frac{d\eta}{dt} = \frac{1}{2\eta}\frac{\partial\eta}{\partial\eta_{ab}}\frac{d\eta_{ab}}{dt} =$$
$$= \frac{1}{2}\eta^{ab}\frac{d\eta_{ab}}{dt} = \varphi_a{}^c\eta_{cb}\eta^{ab} = \varphi_a{}^a, \quad \eta = \det\|\eta_{ab}\|$$

It is clear that the deviator part of the tensor φ will be responsible for the relaxation of the shear deformation [strain] and shear stresses.

In order to avoid any misunderstandings we should note here that the functions $x_*{}^i(\xi^a, t)$ determining the law of change of the SSS for which $\eta_{ab} = x_{*a}{}^i x_{*b}{}^j g_{ij}$, may be non-existent if the functions $\eta_{ab}(\xi^a, t)$ do not make the corresponding curvature tensor equal to zero [2]. This means that the medium will not be able, in this case, to shed all stresses in a purely elastic manner while remaining in three-dimensional space, i.e. the SSS cannot be realized in three-dimensional observer space. This, however, is not important in what follows, since the basic variables, i.e. the MT of SSS η_{ab} are well-defined functions.

When the time is shorter than the characteristic relaxation time, i.e. at times during which Eq.

(2.2) has no chance to develop and is practically identical with Eq. (2.1), the medium behaves as an ordinary elastic body, and for this reason we must retain, for a medium with relaxation, the form of the internal energy density function $U(g_{ab}, \eta_{ab}, S)$ and the equations of state (1.3) of the theory of elasticity. Then identity (1.6) will yield, by virtue of the equations of state, the following expression for the amount of heat generated during the relaxation process:

$$q' = -(\partial U/\partial \eta_{ab})\, d\eta_{ab}/dt = \rho^{-1} p_b{}^a \varphi_a{}^b \tag{2.3}$$

Here we make use of the identity

$$(\partial U/\partial g_{ac})\, g_{cb} + (\partial U/\partial \eta_{ac})\, \eta_{cb} = 0 \tag{2.4}$$

the first relation of (1.3) and Eq. (2.2).

Identity (2.4) follows from the condition that the function U is scalar:

$$U(g_{ab}, \eta_{ab}, S) = U(g_{cd} a_a{}^c a^d{}_b,\ \eta_{cd} a_a{}^c a_b{}^d,\ S) \tag{2.5}$$

where $a_b{}^a$ is the matrix of an arbitrary change of coordinates. In order to obtain (2.4) we must differentiate Eq. (2.5) in $a_b{}^a$ and write $a_b{}^a = \delta_b{}^a$, where $\delta_b{}^a$ is the Kronecker delta.

In order to close the system of equations (1.1)–(1.3), (2.2), (2.3), it remains to specify the expressions for the heat flux vector **q** and the relaxation tensor φ, satisfying the requirement that the gain in entropy

$$\sigma = -T^{-2} (\nabla_i T)\, q^i + T^{-1} p_b{}^a \varphi_a{}^b \tag{2.6}$$

must be positive.

The structure of expression (2.6) implies that the tensor φ plays the role of the thermodynamic flux of the relaxation process, while the stress tensor **p** plays the role of the thermodynamic force. In accordance with general principles of the mechanics of irreversible processes, the thermodynamic fluxes can be naturally expressed in terms of the thermodynamic forces.

3. AN EXPRESSION FOR THE RELAXATION TENSOR IN TERMS OF THE STRESS TENSOR

We shall first assume that no thermodynamic interaction exists between the heat and relaxation processes. We can then adopt for the vector **q** the Fourier heat conduction law (1.5) (in the isotropic case $\varkappa^{ij} = \varkappa g^{ij}$ and the tensor φ will be expressed in terms of tensor **p** only).

We can assume, to a first approximation, the linear dependence between φ and **p**, which can always be written, in the case of an isotropic medium, in the form

$$\varphi_b{}^a = \tau_S{}^{-1} [p_b{}^a - {}^1/_3 p_c{}^c \delta_b{}^a] + \tau_V{}^{-1} [{}^1/_3 p_c{}^c]\, \delta_b{}^a \tag{3.1}$$

when τ_S and τ_V are the characteristic relaxation times of the shear and volume deformations respectively (more accurately, the quantities τ_S and τ_V correspond to characteristic times but are not equal to them, if only because they are of different dimensions). Also,

$$q' = (\rho\tau_V)^{-1} [{}^1/_3 p_c{}^c]^2 + (\rho\tau_S)^{-1} [p_a{}^b - {}^1/_3 p_c{}^c \delta_a{}^b]\, [p_b{}^a - {}^1/_3 p_c{}^c\, \delta_b{}^a] \tag{3.2}$$

The non-negativity of this expression is ensured by the positive form of the quantities τ_S and τ_V.

We can introduce non-linearity into formula (3.1) with the help of the dependence of the quantities τ_S and τ_V on the invariants of the tensor and scalar functions related to the state of the medium. This dependence can, in general, be arbitrary, and has to be found experimentally.

The arbitrary dependence of φ on $\mathbf{p}$ satisfying the demand that q' be positive can always be written, in the isotropic case, in the form

$$\varphi_a{}^b = A_1 p_a{}^b + A_2 p_a{}^c p_c{}^d p_d{}^b \tag{3.3}$$

where A_1 and A_2 are positive invariants of the stress tensor $\mathbf{p}$ and of the other scalar parameters of the medium. Here we also have

$$q' = \rho^{-1}\,[A_1 p_a{}^b p_b{}^a + A_2 p_a{}^c p_c{}^b p_b{}^d p_d{}^a] \tag{3.4}$$

4. AN EXPRESSION FOR THE RELAXATION TENSOR IN TERMS OF MT OF THE SSS

Since the stress tensor is expressed in terms of the tensor η, it follows that (3.3) gives, in fact, the expression for the relaxation tensor φ in terms of the MT of the SSS η. We shall also consider here the problem of the possibility of expressing φ directly in terms of η. Since the tensors φ, $\mathbf{p}$ and η are expressed in terms of each other in an isotropic manner, it follows that they are coaxial (this agrees with the experimental fact according to which the principal axes of the stress and strain tensors do not vary in the relaxation process). Then

$$q' = \frac{1}{\rho}\sum_{n=1}^{3} p_n\varphi_n = \frac{1}{6\rho}\Big[\sum_{\substack{n=1\\m=1}}^{3}(p_n - p_m)(\varphi_n - \varphi_m) + 2\Big(\sum_{n=1}^{3} p_n\Big)\Big(\sum_{m=1}^{3}\varphi_m\Big)\Big]$$

$$(\det\|\, p_b{}^a - p_n\delta_b{}^a \,\| = 0,\quad \det\|\,\varphi_b{}^a - \varphi_n\delta_b{}^a\,\| = 0) \tag{4.1}$$

where p_n and φ_n are the eigenvalues of the tensors $\mathbf{p}$ and φ, respectively, i.e. are the roots of the equations given above in brackets.

The non-negativity of the quantity q' under the conditions of the direct dependence of φ on η, is ensured as a result of the inequality

$$(p_n - p_m)/((k_n)^2 - (k_m)^2) > 0 \tag{4.2}$$

which was obtained [3] as the necessary condition of correctness of the Cauchy problem for the system of equations of the theory of elasticity. Here $k_n{}^2$ are the eigenvalues of the tensors φ, i.e. are the roots of the equation

$$\det\|\,\eta^{ac}g_{cb} - k_n{}^2\delta_b{}^a\,\| = 0$$

From the physical point of view inequality (4.2) is natural, since it means that the force appearing during the shearing action is directed against it.

If no relaxation of the density in the SSS appears in the medium (no pressure relaxation), we can use the deviator of any monotonically increasing function $f(x)$ applied to the matrix $\eta^{ac}g_{cd}$ as the relaxation tensor φ:

$$\varphi_a{}^b = \operatorname{dev}\,[f\,(\eta^{bc}g_{ca})] \tag{4.3}$$

The function f of the matrix A is usually understood as $f(A) = \Sigma c_n A^n$, where $f(x) = \Sigma c_n x^n$ is any expansion of the function $f(x)$ in powers of x. If

$$\eta^{ac}g_{cb} = \sum_{n=1}^{3} k_n{}^2 w_n{}^a w_{nb}$$

is the expansion of the matrix η^{ab} in the characteristic basis w_{na} (n is the number of the eigenvector)

$$w_n{}^a w_{ma} = \delta_{nm}, \quad \sum_{n=1}^{3} w_n{}^a w_{nb} = \delta_b{}^a, \quad w_n{}^a = g^{ab} w_{nb}$$

then

$$f(\eta^{ac} g_{cb}) = \sum_{n=1}^{3} f(k_n{}^2)\, w_n{}^a w_{nb}$$

We also have

$$q' = \frac{1}{6\rho} \sum_{\substack{n=1\\m=1}}^{3} (p_n - p_m)(f(k_n{}^2) - f(k_m{}^2)) \tag{4.4}$$

It is clear that the non-negative form of the quantity q' follows from inequality (4.2), provided that $f(x)$ is a monotonically increasing function of its argument.

Step 1. In particular, we can take $f(x) = x/\tau_1$ as $f(x)$, in which case

$$\varphi_b{}^a = \tau_1{}^{-1} [\eta^{ac} g_{cb} - {}^1/_3 \delta_b{}^a \eta^{cd} g_{cd}] \tag{4.5}$$

Here, as elsewhere, τ_1 is the characteristic relaxation time, which can be a function of the scalar parameters of the medium.

The relaxation equation (2.2), (4.5) was given in [3]. To match the formulas derived above with the formulas in [3], we must note that the effective strain tensor g_{*ij} adopted as the basic relaxation variable in [3], can be regarded as the expression for the MT of the SSS η in Eulerian space

$$g_{*ij} = \xi^a{}_i \xi^b{}_j \eta_{ab}, \quad \eta_{ab} = x^i{}_a x_b{}^j g_{*ij} \tag{4.6}$$

Then

$$d\eta_{ab}/dt = x_a{}^i x_b{}^j dg_{*ij}/dt + g_{*ij} x_b{}^j \nabla_a v^i + g_{*ij} x_a{}^i \nabla_b v^j \tag{4.7}$$

where we have used the relation $dx_a{}^i/dt = \nabla_a v^i$.

The equation for g_{*ij} is obtained from (4.7) and (2.2) by multiplying by $\xi_i{}^a$ over the indices a and b:

$$dg_{*ij}/dt + g_{*ik} \nabla_j v^k + g_{*jk} \nabla_i v^k = g_{*ik} \varphi_j{}^k \tag{4.8}$$

After substituting expression (4.5) into relation (4.3) and replacing the variable τ_1 by the variable τ from [3] using the formula $\tau_1 = 1/3\tau\eta^{cd} g_{cd}$, we can confirm that the right-hand side of the first equation of (4.8) as identical with the right-hand side of the corresponding equation of [3]. Here we must also take into account the fact that the derivatives $\rho_{\varepsilon_{ij}}$ of the density $\rho = \rho^0 \sqrt{g_*}$ ($g_* = \det\|g_{*ij}\|$ introduced in [3] can be expressed in terms of the strain tensor $\varepsilon_{ij} = \frac{1}{2}(g_{ij} - g_{*ij})$ as follows:

$$\frac{1}{\rho} \rho_{\varepsilon_{ij}} = \frac{1}{\rho^0 \sqrt{g^*}} \frac{\partial \rho^0 \sqrt{g_*}}{\partial \varepsilon_{ij}} = \frac{1}{2g_*} \frac{\partial g_*}{\partial g_{*kl}} \frac{\partial g_{*kl}}{\partial \varepsilon_{ij}} =$$
$$= \frac{1}{2} g_*{}^{kl} \frac{\partial}{\partial \varepsilon_{ij}} (g_{kl} - 2\varepsilon_{kl}) = -g_*{}^{ij}$$

In order to avoid confusion, we also note that the formula for the density $\rho = \rho^0 \sqrt{g_*}$, used in [3, 4] is invalid within the framework of the present paper and must be replaced by the usual formula

$$\rho = \rho^0 \sqrt{g^0}/\sqrt{g} = \rho^0 \sqrt{g^0} \sqrt{g^*}/\sqrt{\eta \det \| g_{ij} \|}$$

Step 2. If we take as $f(x)$ the monotonically increasing function $f(x) = -(\tau_2 x)^{-1}$, then

$$\varphi_b{}^a = -\tau_2^{-1}\,[g^{ac}\eta_{cb} - {}^1/_3\delta_b{}^a g^{cd}\eta_{cd}] \tag{4.9}$$

Step 3. From the general theory of tensor functions it follows that if the tensor φ can be expressed in terms of tensor η (when there is no volume relaxation we have $\varphi_a{}^a = 0$) using relation (4.3), then it can always be written as follows:

$$\varphi_b{}^a = \tau_1^{-1}\,[\eta^{ac}g_{cb} - {}^1/_3\eta^{cd}g_{cd}\delta_b{}^a] - \\ - \tau_2^{-1}\,[g^{ac}\eta_{cb} - {}^1/_3 g^{cd}\eta_{cd}\delta_b{}^a] \tag{4.10}$$

where τ_1 and τ_2 are positive scalar functions of arbitrary scalar arguments.

Step 4. Irrespective of the fact that any dependence of φ on η satisfying the requirement that q' be non-negative can be written in the form (4.10), we may find that another expression may be more suitable when approximating the empirical data. For example, in [4] the relaxation tensor φ was considered, expressed in terms of the Hencky strain tensor

$$D_b = {}^1/_2 \ln(\eta^{ac}g_{cb}) = \sum_{n=1}^{3} \ln k_n w_n{}^a w_{nb}$$

which corresponds to the choice of $f(x) = \tau^{-1}\ln x$. We then have

$$\varphi_b{}^a = \tau^{-1}\,[D_b{}^a - {}^1/_3\delta_b{}^a D_c{}^c] \tag{4.11}$$

We can also establish a relation between formulas (4.11) and those in [4] using relations (4.6)–(4.8).

In order to find the volume relaxation in strain terms, we must specify the expression for the trace of the tensor φ in terms of invariants of the tensor η. However, the necessary conditions for the correctness of the Cauchy problem do not contain the condition for the spherical part of the stress tensor analogous to (4.2) for shear stresses, and it becomes difficult to ensure that q' is non-negative. Therefore, it follows that within the framework of the approach adopted here there are no general equations containing the general relaxation expressed in deformation terms. However, in certain special cases we may find, by virtue of specific features of the functions $U(g_{ab}, \eta_{ab}, S)$, that some of the expressions $p_a{}^a\eta^{cb}g_{cb}$, $p_a{}^a g_{cb}\eta_{cb}$ or $p_a{}^a D^b{}_b$ are positive definite. In this case we can substitute the corresponding spherical parts into formulas (4.5), (4.9)–(4.11).

The question of which expression of the tensor φ leads to the simplest approximation of the empirical date remains open.

5. THE RELAXATION EQUATIONS IN AN ANISOTROPIC MEDIUM

We can take as additional tensor arguments of the functions $U(g_{ab}, S)$ for an anisotropic medium in the general state the triplet of vectors a_{na} (n is the number of the vector and a is its projection), specifying the orientation of selected directions in the medium

$$U = U\,(g_{ab},\ a_{na},\ S) \tag{5.1}$$

In the case of anisotropic media with a high degree of symmetry (cubic, hexagonal, etc.) the vector triplet $\mathbf{a}_n$ is not unique, but it does not matter in what follows.

We can ssume, without loss of generality, that the vectors $\mathbf{a}_n$ are orthonormed in the metric of the SSS:

$$a_{na}a_{mb}\eta^{ab} = \delta_{nm}, \quad a_n{}^a = a_{nb}\eta^{ab}$$
$$\eta_{ab} = a_{na}a_b{}^n, \quad a_n{}^a a_b{}^n = \delta_b{}^a \tag{5.2}$$

Here and henceforth, irrespective of the fact that the index n has no tensor properties, we apply to it the rule of summation over repeated upper and lower indices, and juggle them with the help of the Kronecker delta δ. The vectors $\mathbf{a}_n$ refer to the stress-free state, and their indices are therefore juggled with the help of the metric η of the stress-free state.

In the theory of elasticity of an anisotropic medium the vectors $\mathbf{a}_n$ do not depend on time: $da_{na}/dt = 0$.

Furthermore, in accordance with the scheme proposed above, in constructing the theory of a relaxing anisotropic medium we allow the vectors $\mathbf{a}_n$ to vary with time, while the function $U(g_{ab}, a_{na}, S)$ and equations of state (1.3) remain unchanged. Also, the following expression results from the requirement that relation (1.6) must be an identity:

$$q' = -\partial U/\partial a_{na} da_{na}/dt \tag{5.3}$$

It is natural to denote the quantity $\rho\partial U/\partial a_{na}$ by p.

Using the identity

$$2\partial U/\partial g_{ac}g_{bc} + \partial U/\partial a_{na}a_{nb} = 0 \tag{5.4}$$

[which is obtained in the same manner as identity (2.4)], we can find that

$$p^{na} = p_b{}^a a^{nb} \quad (p^{na} = -\rho\partial U/\partial a_{na})$$

In order to avoid any confusion we note that $p^{na} \neq p^{ab}a_b{}^n$, since $p^{ab} = p_c{}^a g^{cb}$, $a_{bn} = a^{nc}g_{cb}$ and $g^{cb}\eta_{ac} \neq \delta_a{}^b$. Also $p_b{}^a = p^{na}a_{nb}$, but $p^{ab} \neq p^{na}a_n{}^b$.

We can write, without loss of generality, the relaxation equations of an anisotropic medium, in the form

$$da_{na}/dt = \varphi_a{}^b a_{nb} \tag{5.5}$$

For completeness of the presentation we note the validity of the formulas

$$da_n{}^a/dt = -\varphi_b{}^a a_n{}^b$$
$$d\eta_{ab}/dt = \varphi_a{}^c\eta_{bc} + \varphi_b{}^c\eta_{ac}$$

which follow from relations (5.2) and (5.5).

We will denote the relaxation tensor φ using the same letter as in the isotropic case, since both these tensors have much in common. In particular, the spherical part of the tensor φ is responsible, as before, for the change in the density of the stress-free state:

$$\rho_* d\rho_*^{-1}/dt = {}^1/_2\eta^{ab}d\eta_{ab}/dt =$$
$$= {}^1/_2\eta^{ab}[\varphi_a{}^c\eta_{bc} + \varphi_b{}^c\eta_{ac}] = \varphi_a{}^a$$

Using the identity (5.4), we can rewrite expression (5.3) for q' as follows:

$$q' = \rho^{-1}p_b{}^a\varphi_a{}^b \tag{5.6}$$

Relation (5.6) has the same form as (2.3), therefore everything that was stated about the tensor in

Secs 2 and 3, holds for $\boldsymbol{\varphi}$. However, the relation between $\boldsymbol{\varphi}$ and $\mathbf{p}$ does not now have to be isotropic. In the linear case it will take the form

$$\varphi_a{}^b = \tau_{a\cdot c\cdot}^{\cdot b\cdot d} p_d{}^c \tag{5.7}$$

where the tetravalent tensor $\boldsymbol{\tau}$ can naturally be regarded as the tensor of characteristic relaxation times (not in a literal sense, since the dimensions of $\boldsymbol{\tau}$ again are not the dimensions of time).

By virtue of the second law of thermodynamics the tensor $\boldsymbol{\tau}$ must be positive definite, since

$$q' = \rho^{-1}\tau_{abcd}p^{ab}p^{cd}$$

and symmetrical by virtue of Onsager's principle

$$\tau_{abcd} = \tau_{cdab}, \quad \tau_{abcd} = \tau_{badc}$$

In the general, non-linear case, the tensor $\boldsymbol{\tau}$ can be an arbitrary function of any tensor and scalar parameters of the medium.

The analogues of the inequalities (4.2) for the anisotropic case are not known, and we therefore cannot express the tensor $\boldsymbol{\tau}$ directly in terms of $\boldsymbol{\eta}$ or $\mathbf{a}_n$.

We can again use the Fourier law of heat conduction (1.5) as the expression for the heat flux vector $\mathbf{q}$.

6. INTERACTION BETWEEN THE THERMAL AND RELAXATION PROCESSES

We shall now assume that thermodynamic interaction occurs between the relaxation and thermal processes. The equations connecting the thermodynamic fluxes and forces must, in this case, ensure that the total increase in entropy is positive

$$\sigma = -[T^{-2}\nabla_a T]\, q^a + T^{-1}p_b{}^a\varphi_a{}^b \tag{6.1}$$

Here the quantities $\mathbf{q}$ and $\boldsymbol{\varphi}$ are thermodynamic fluxes and the quantities $-T^{-2}\nabla T$ and $\mathbf{p}$ represent the thermodynamic forces. If the relation between them is linear, it can be expressed in the form

$$q^a = -\varkappa^{ab}\nabla_b T + \alpha^{ab\cdot}_{\cdot\cdot c}p_b{}^c$$
$$\varphi_b{}^a = \tau_{b\cdot c\cdot}^{\cdot a\cdot d}p_d{}^c - \beta^{ca\cdot}_{\cdot\cdot b}T^{-1}\nabla_c T$$

where, by virtue of Onsager's principle, in spite of the symmetrical form of τ and $\varkappa$, it is necessary that

$$\alpha^{ab\cdot}_{\cdot\cdot c} = \beta^{ab\cdot}_{\cdot\cdot c}, \quad \alpha^{abc} = \alpha^{acb}$$

In the general case, the tensors $\boldsymbol{\tau}$, $\boldsymbol{\varkappa}$ and $\boldsymbol{\alpha}$ can be arbitrary functions of the parameters of the medium, ensuring that expression (6.1) for the gain in entropy is non-negative.

7. THE ASYMPTOTIC LYAPUNOV STABILITY OF THE SSS

In purely relaxation isentropic adiabatic and isothermal processes the asymptotic Lyapunov stability of the SSS follows from the fact that the quantity σ is positive.

Let the external heat source r maintain, in a spatially homogeneous relaxation process in a medium at rest, a constant entropy $S(\xi^a, t) = S^0$. Then from the second and third equation of (1.1) it follows that $r = -\rho q'$ and

$$dU/dt = -q' < 0$$

It is clear that the function U decreases monotonically in this process and, acting as the Lyapunov function, tends to its minimum, which is realized at a corresponding fixed value of the entropy S^0 of the SSS.

In the adiabatic process ($r = 0$), the function $U = U^0 = \text{const}$, and the part of the Lyapunov function is played by the entropy (with opposite sign) as the function of g_{ab}, η_{ab} (a_{na}) and U^0 which, by virtue of the last equation of (1.1), also attains its extremum in the SSS, the SSS corresponding to the fixed value of the energy U^0.

In the isothermal, homogeneous and purely relaxation process the volume source of heat r keeps the temperature $T = T^0$ constant. From relation (1.6) it follows that the part of the Lyapunov function is played in this case by the free energy $F = U - T^0 S$, whose minimum is also realized in the corresponding SSS.

It is clear that the list of particular processes with an asymptotically stable final state can be extended.

In conclusion we note that linearization of the relations given above leads to the usual Maxwell's theory of relaxation.

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